

Fermionic vacuum polarization by a cosmic string in de Sitter spacetime

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June 2, 2010

Abstract

We investigate the fermionic condensate and the vacuum expectation value of the energy-momentum tensor for a massive spinor field in the geometry of a straight cosmic string on background of de Sitter spacetime. By using the Abel-Plana summation formula, we explicitly extract from the expectation values the contribution associated with purely de Sitter space, remaining the expectation values induced by the cosmic string. The latter presents information about de Sitter gravity as well. Because the investigation of the fermionic quantum fluctuations in de Sitter space have been investigated in literature, here we are mainly interested in the cosmic string-induced contributions. For a massless field, the fermionic condensate vanishes and the presence of the string does not break chiral symmetry of the massless theory. Unlike to the case of a scalar field, for a massive fermionic field the vacuum expectation value of the energy-momentum tensor is diagonal and the axial and radial stresses are equal to the energy density. At large distances from the string the behavior of the string-induced parts in the vacuum densities is damping oscillatory with the amplitude decaying as the inverse fourth power of the distance. This is in contrast to the case of flat spacetime, in which the string-induced vacuum densities for a massive field decay exponentially with distance from the string. In the limit of the large curvature radius of de Sitter space we recover the results for a cosmic string in flat spacetime.

PACS numbers: 03.70.+k, 98.80.Cq, 11.27.+d

1 Introduction

Different types of topological objects may have been formed in the early universe after Planck time by the vacuum phase transition [1]. Depending on the topology of the vacuum manifold these are domain walls, strings, monopoles and textures. Among them the cosmic strings are of special interest. Although the recent observational data on the cosmic microwave background radiation have ruled out cosmic strings as the primary source for primordial density perturbations, they are still candidates for the generation of a number of interesting physical effects such

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as gravitational lensing, cosmic microwave background non-gaussianities, the emission of gravitational waves and high-energy cosmic rays (see, for instance, [2]). More recently, cosmic strings attract a renewed interest partly because a variant of their formation mechanism is proposed in the framework of brane inflation [3].

In the simplest theoretical model describing the infinite straight cosmic string the spacetime is locally flat except on the string where it has a delta shaped Riemann curvature tensor. In quantum field theory the corresponding non-trivial topology induces non-zero vacuum expectation values for physical observables. Explicit calculations for the geometry of a single cosmic string have been done for different fields [4]-[29]. Vacuum polarization effects by higher-dimensional composite topological defects constituted by a cosmic string and global monopole are investigated in Refs. [30] for scalar and fermionic fields. Another type of vacuum polarization arises when boundaries are present. The imposed boundary conditions on quantum fields alter the zero-point fluctuations spectrum and result in additional shifts in the vacuum expectation values of physical quantities. In Ref. [31], we have studied both types of sources for the polarization of the scalar vacuum, namely, a cylindrical boundary and a cosmic string, assuming that the boundary is coaxial with the string and that on this surface the scalar field obeys Robin boundary condition. The polarization of the electromagnetic vacuum by a conducting cylindrical shell in the cosmic string spacetime is investigated in [32]. The case of the fermionic field with bag boundary condition on the cylindrical surface is discussed in [33].

Many of treatments of quantum fields around a cosmic string deal mainly with the flat background geometry. Quantum effects for a scalar field produced by a string on curved backgrounds have been investigated in [10] for special values of planar angle deficit when the corresponding two-point functions can be constructed by making use of the method of images. In a previous paper [34] we have investigated the vacuum polarization effect associated with a quantum massive scalar field in de Sitter (dS) spacetime in the presence of a cosmic string (for cosmic strings in background of dS spacetime see [35]-[38]). It has been shown that the corresponding gravitational field essentially changes the behavior of the vacuum densities at distances from the string larger than the dS curvature radius. Depending on the curvature radius of de Sitter spacetime, two regimes are realized with monotonic and oscillatory behavior of the vacuum expectation values at large distances. Another interesting feature due to the background gravitational field is the appearance of non-zero off-diagonal component of the energy-momentum tensor which corresponds to the energy flux along the radial direction (see, also, [10]; a similar effect induced by plane boundaries in dS spacetime has been observed in [39]). In [40] we have considered the influence of a cosmic string to the power-spectrum of quantum fluctuations for a scalar field in dS spacetime. The dependence of the power-spectrum on the distance from the string is oscillatory with the period having the order of the wavelength for the perturbation. One-loop topological quantum effects for scalar and fermionic fields induced by the toroidal compactification of spatial dimensions in dS spacetime have been recently considered in [41, 42].

In the present paper we provide the results of the investigation for the fermionic vacuum polarization by a cosmic string in dS spacetime. dS spacetime is the maximally symmetric solution of the Einstein equations with a positive cosmological constant and due to its high symmetry numerous physical problems are exactly solvable on this background. A better understanding of physical effects in this background could serve as a handle to deal with more complicated geometries. In most inflationary models an approximately dS spacetime is employed to solve a number of problems in standard cosmology [43]. More recently astronomical observations of high redshift supernovae, galaxy clusters and cosmic microwave background [44] indicate that at the present epoch the universe is accelerating and can be well approximated by a world with a positive cosmological constant. If the universe would accelerate indefinitely, the standard cosmology would lead to an asymptotic dS universe. In addition to the above, an interesting topic

which has received increasing attention is related to string-theoretical models of dS spacetime and inflation. Recently a number of constructions of metastable dS vacua within the framework of string theory are discussed (see, for instance, [45] and references therein).

The results obtained here can be used, in particular, for the investigation of the effects of the quantum fluctuations induced by the string in the inflationary phase. Though the cosmic strings produced in phase transitions before or during early stages of inflation would have been drastically diluted by the expansion, the formation of defects during inflation can be triggered by a coupling of the symmetry breaking field to the inflaton field or to the curvature of the background spacetime (see [1]). Cosmic strings can also be continuously created during inflation by quantum-mechanical tunnelling [46]. Another class of models to which the results of the present paper are applicable corresponds to string-driven inflation where the cosmological expansion is driven entirely by the string energy [47]. The problem under consideration is also of separate interest as an example with gravitational and topological polarizations of the fermionic vacuum, where all calculations can be performed in a closed form.

The plan of the paper is the following. In the next section, the geometry under consideration is described and a complete set of solutions to Dirac equation is constructed. In section 3 we evaluate the fermionic condensate by using the mode-summation method. The string induced part is explicitly extracted and its behavior in the asymptotic regions of the parameters are investigated. The vacuum expectation value of the energy-momentum tensor is considered in section 4. The main results of the paper are summarized in section 5. The Appendix A contains some technical details of the obtainment of the vacuum expectation values. In Appendix B we consider the expectation values in dS spacetime when the string is absent.

2 Fermionic eigenfunctions

The main objective of the present section is to obtain the complete set of solutions of Dirac equation in dS spacetime in presence of an infinitely long straight cosmic string. In order to do that, we write the corresponding line element in cylindrical coordinates, having the linear defect along the z -axis:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - e^{2t/\alpha} (dr^2 + r^2 d\phi^2 + dz^2), \quad (2.1)$$

where $r \geq 0$, $-\infty < z < \infty$, $0 \leq \phi \leq \phi_0 \leq 2\pi$ and the spatial points (r, ϕ, z) and $(r, \phi + \phi_0, z)$ are to be identified. The parameter α in (2.1) is related with the cosmological constant Λ by the formula $\alpha = \sqrt{3/\Lambda}$. Making use of the coordinate transformation

$$\begin{aligned} t &= t_s - \alpha \ln f(r_s), \quad r = r_s f(r_s) e^{-t_s/\alpha} \sin \theta, \\ z &= r_s f(r_s) e^{-t_s/\alpha} \cos \theta, \quad \phi = \phi, \end{aligned} \quad (2.2)$$

with $f(r_s) = 1/\sqrt{1 - r_s^2/\alpha^2}$, the line element (2.1) is written in static form

$$ds^2 = f^{-2}(r_s) dt_s^2 - f^2(r_s) dr_s^2 - r_s^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (2.3)$$

Rescaling the angular variable in accordance with $\varphi = 2\pi\phi/\phi_0$, we transform the metric corresponding to (2.3) into the form previously discussed in [37]. In this paper it is shown that to leading order in the gravitational coupling the effect of the vortex on de Sitter spacetime is to create a deficit angle in the metric (2.3).

The dynamics of a massive spinor field on a curved spacetime are described by Dirac equation

$$i\gamma^\mu \nabla_\mu \psi - m\psi = 0, \quad \nabla_\mu = \partial_\mu + \Gamma_\mu, \quad (2.4)$$

where γ^μ are the Dirac matrices in curved spacetime and Γ_μ is the spin connection. They are given in terms of the flat space Dirac matrices $\gamma^{(a)}$ by the relations

$$\gamma^\mu = e_{(a)}^\mu \gamma^{(a)}, \quad \Gamma_\mu = \frac{1}{4} \gamma^{(a)} \gamma^{(b)} e_{(a)}^\nu e_{(b)\nu;\mu}, \quad (2.5)$$

where the semicolon means the standard covariant derivative for vector fields. In (2.5), $e_{(a)}^\mu$ is the tetrad field satisfying the relation $e_{(a)}^\mu e_{(b)\mu} \eta^{ab} = g^{\mu\nu}$, with η^{ab} being the Minkowski spacetime metric tensor.

In the discussion below the flat space Dirac matrices will be taken in the standard form [48]

$$\gamma^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^{(a)} = \begin{pmatrix} 0 & \sigma_a \\ -\sigma_a & 0 \end{pmatrix}, \quad a = 1, 2, 3, \quad (2.6)$$

with $\sigma_1, \sigma_2, \sigma_3$ being the Pauli matrices. The tetrad fields corresponding to line element (2.1) may have the form

$$e_{(a)}^\mu = e^{-t/\alpha} \begin{pmatrix} e^{t/\alpha} & 0 & 0 & 0 \\ 0 & \cos(q\phi) & -\sin(q\phi)/r & 0 \\ 0 & \sin(q\phi) & \cos(q\phi)/r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (2.7)$$

where

$$q = 2\pi/\phi_0 \geq 1. \quad (2.8)$$

For the curved space gamma matrices this leads to the formulae

$$\gamma^0 = \gamma^{(0)}, \quad \gamma^l = e^{-t/\alpha} \begin{pmatrix} 0 & \beta^l \\ -\beta^l & 0 \end{pmatrix}, \quad (2.9)$$

with the 2×2 matrices

$$\beta^1 = \begin{pmatrix} 0 & e^{-iq\phi} \\ e^{iq\phi} & 0 \end{pmatrix}, \quad \beta^2 = -\frac{i}{r} \begin{pmatrix} 0 & e^{-iq\phi} \\ -e^{iq\phi} & 0 \end{pmatrix}, \quad \beta^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.10)$$

In (2.9) and below the index l runs over values 1, 2, 3. For the components of the spin connection we find

$$\Gamma_0 = 0, \quad \Gamma_l = -\frac{1}{2\alpha} \gamma^0 \gamma_l + \frac{1-q}{2} \gamma^{(1)} \gamma^{(2)} \delta_l^2. \quad (2.11)$$

This leads to the following expression for the combination appearing in Dirac equation:

$$\gamma^\mu \Gamma_\mu = \frac{3\gamma^0}{2\alpha} + \frac{1-q}{2r} \gamma^1. \quad (2.12)$$

For further analysis we shall adopt the conformal time τ defined by $\tau = -\alpha e^{t/\alpha}$, $-\infty < \tau < 0$.

Decomposing the bispinor ψ into the upper and lower two-component spinors, denoted by φ and χ respectively, Dirac equation is written in the form of two coupled first order differential equations:

$$\hat{D}_+ \varphi + \left(\beta^l \partial_l + \frac{1-q}{2r} \beta^1 \right) \chi = 0, \quad (2.13)$$

$$\hat{D}_- \chi + \left(\beta^l \partial_l + \frac{1-q}{2r} \beta^1 \right) \varphi = 0. \quad (2.14)$$

Here we have defined the operators

$$\hat{D}_{\pm} = \partial_{\tau} - \frac{1}{\tau} \left(\frac{3}{2} \pm im\alpha \right). \quad (2.15)$$

Note that we are still working in the coordinate system corresponding to line element (2.1).

By using the properties of the matrices (2.10), we obtain the following second order differential equation for the upper component of the bispinor:

$$\left[\partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_{\phi}^2 + \partial_z^2 + \frac{q-1}{r} \beta^1 \beta^2 \partial_{\phi} - \frac{(q-1)^2}{4r^2} - \hat{D}_+ \hat{D}_- \right] \varphi = 0. \quad (2.16)$$

Similar equation is obtained for the lower component χ with $\hat{D}_- \hat{D}_+$ instead of $\hat{D}_+ \hat{D}_-$. Since the matrix $\beta^1 \beta^2$ is diagonal, from equation (2.16) it follows that if we write

$$\varphi = \begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{pmatrix}, \quad (2.17)$$

then the equations for the upper and lower components are decomposed:

$$\left[\partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_{\phi}^2 + \partial_z^2 - (-1)^a i \frac{q-1}{r^2} \partial_{\phi} - \frac{(q-1)^2}{4r^2} - \hat{D}_+ \hat{D}_- \right] \varphi^{(a)} = 0. \quad (2.18)$$

with $a = 1, 2$.

The solution of equation (2.18) can be presented in the form

$$\varphi^{(a)} = f^{(a)}(\tau, r) e^{i(qn_a \phi + kz)}, \quad (2.19)$$

with $n_a = 0, \pm 1, \pm 2, \dots$, $-\infty < k < \infty$, and with the equation for the function $f^{(a)}(\tau, r)$:

$$\left(\partial_r^2 + \frac{1}{r} \partial_r - \frac{\beta_a^2}{r^2} - k^2 - \hat{D}_+ \hat{D}_- \right) f^{(a)}(\tau, r) = 0. \quad (2.20)$$

Here we have introduced the notation

$$\beta_a = |qn_a - (-1)^a (q-1)/2|. \quad (2.21)$$

From (2.20) it follows that the time and radial coordinate dependences of the function $f^{(a)}(\tau, r)$ can be separated with the solution

$$f^{(a)}(\tau, r) = T^{(a)}(\tau) J_{\beta_a}(\lambda r), \quad (2.22)$$

where $0 \leq \lambda < \infty$ and $J_{\nu}(x)$ is the Bessel function.

For the function $T^{(a)}(\tau)$ in (2.22) one finds the equation

$$T^{(a)''}(\tau) - \frac{3}{\tau} T^{(a)'}(\tau) + \left(\lambda^2 + k^2 + \frac{m^2 \alpha^2 + im\alpha + 15/4}{\tau^2} \right) T^{(a)}(\tau) = 0. \quad (2.23)$$

The general solution for this equation has the form $T^{(a)}(\tau) = \tau^2 T_{a\varphi}(\tau)$ with

$$T_{a\varphi}(\tau) = \sum_{l=1,2} C_{a\varphi}^{(l)} H_{1/2-im\alpha}^{(l)}(\gamma\eta), \quad (2.24)$$

where

$$\eta = |\tau|, \quad \gamma = \sqrt{\lambda^2 + k^2}, \quad (2.25)$$

and $H_\nu^{(l)}(x)$ are the Hankel functions. Hence, for the upper and lower components of the spinor φ we have the solutions

$$\varphi^{(a)} = \eta^2 T_{a\varphi}(\tau) J_{\beta_a}(\lambda r) e^{i(qn_a\phi + kz)}. \quad (2.26)$$

Note that the following relation takes place:

$$\hat{D}_+ [\tau^2 T_{a\varphi}(\tau)] = -\gamma \eta^2 \sum_{l=1,2} C_{a\varphi}^{(l)} H_{-1/2-im\alpha}^{(l)}(\gamma\eta). \quad (2.27)$$

In a similar way, it can be seen that if we write the lower component of the bispinor as

$$\chi = \begin{pmatrix} \chi^{(1)} \\ \chi^{(2)} \end{pmatrix}, \quad (2.28)$$

then for the separate functions one has the solution

$$\chi^{(a)} = \tau^2 T_{a\chi}(\tau) J_{\tilde{\beta}_a}(\lambda r) e^{i(q\tilde{n}_a\phi + kz)}, \quad a = 1, 2. \quad (2.29)$$

Here $\tilde{n}_a = 0, \pm 1, \pm 2, \dots$, $\tilde{\beta}_a = |q\tilde{n}_a - (-1)^a(q-1)/2|$, and we have defined the functions

$$T_{a\chi}(\tau) = \sum_{l=1,2} C_{a\chi}^{(l)} H_{-1/2-im\alpha}^{(l)}(\gamma\eta). \quad (2.30)$$

For these functions one has the relations

$$\hat{D}_- [\tau^2 T_{a\chi}(\tau)] = \gamma \eta^2 \sum_{l=1,2} C_{a\chi}^{(l)} H_{1/2-im\alpha}^{(l)}(\gamma\eta). \quad (2.31)$$

Now, from equations (2.13) and (2.14) we obtain relations between the parameters in the solutions for the functions $\varphi^{(a)}$ and $\chi^{(a)}$:

$$n_2 = n_1 + 1, \quad \beta_2 = \beta_1 + \epsilon_{n_1}, \quad \tilde{n}_a = n_a, \quad \tilde{\beta}_a = \beta_a, \quad (2.32)$$

and for the coefficients in the linear combinations of the Hankel functions:

$$\begin{aligned} \gamma C_{1\chi}^{(l)} &= -\lambda \epsilon_{n_1} C_{2\varphi}^{(l)} - ik C_{1\varphi}^{(l)}, \\ \gamma C_{2\chi}^{(l)} &= \lambda \epsilon_{n_1} C_{1\varphi}^{(l)} + ik C_{2\varphi}^{(l)}, \end{aligned} \quad (2.33)$$

with $l = 1, 2$, and $\epsilon_n = 1$ for $n \geq 0$ and $\epsilon_n = -1$ for $n < 0$.

Different choices of the coefficients in the linear combination (2.24) correspond to different choices of the vacuum state. We will consider a dS invariant Bunch-Davies vacuum (also known as the euclidean vacuum) [49] for which the coefficient for the part containing the function $H_{1/2-im\alpha}^{(2)}(\gamma\eta)$ is zero: $C_{a\varphi}^{(2)} = 0$. The Bunch-Davies vacuum reduces to the standard Minkowski vacuum when dS curvature radius is taken to infinity. From relations (2.33) it follows that $C_{a\chi}^{(2)} = 0$. Note that with this choice, the solution under consideration reduces to the standard positive frequency solutions in the limit $\eta \rightarrow \infty$. In this sense, we will refer to the corresponding solutions in dS spacetime as positive frequency solutions (the same for the negative frequency solutions, see below). Hence, for the positive frequency modes we have

$$\psi_\sigma^{(+)}(x) = \frac{\eta^2}{\gamma} C_{1\varphi}^{(1)} e^{i(qn\phi + kz)} \begin{pmatrix} \gamma H_{1/2-im\alpha}^{(1)}(\gamma\eta) J_{\beta_1}(\lambda r) \\ \gamma C_\varphi^{(1)} H_{1/2-im\alpha}^{(1)}(\gamma\eta) J_{\beta_2}(\lambda r) e^{iq\phi} \\ (\lambda \epsilon_n C_\varphi^{(1)} + ik) H_{-1/2-im\alpha}^{(1)}(\gamma\eta) J_{\beta_1}(\lambda r) \\ -(\lambda \epsilon_n + ik C_\varphi^{(1)}) H_{-1/2-im\alpha}^{(1)}(\gamma\eta) J_{\beta_2}(\lambda r) e^{iq\phi} \end{pmatrix}, \quad (2.34)$$

where $n = 0, \pm 1, \pm 2, \dots$, $C_\varphi^{(1)} = C_{2\varphi}^{(1)}/C_{1\varphi}^{(1)}$, and

$$\beta_1 = |q(n + 1/2) - 1/2|, \quad \beta_2 = \beta_1 + \epsilon_n. \quad (2.35)$$

Here the index σ for the eigenfunctions stands for the set of quantum numbers specifying the solution. This set will be specified below.

For the further specification of the eigenfunctions, following [50], we define the operator

$$\hat{S} = \gamma^{-1} \Sigma^l \hat{p}_l, \quad (2.36)$$

with

$$\hat{p}_1 = -i\partial_r, \quad \hat{p}_2 = -i\partial_\phi + \frac{q-1}{2}\Sigma^3, \quad \hat{p}_3 = -i\partial_z \quad (2.37)$$

and

$$\Sigma^l = \begin{pmatrix} \beta^l & 0 \\ 0 & \beta^l \end{pmatrix}. \quad (2.38)$$

Imposing that the solutions $\psi_\sigma^{(+)}$ obey the condition

$$\hat{S}\psi_\sigma^{(+)} = s\psi_\sigma^{(+)}, \quad (2.39)$$

for the eigenvalues s and for the coefficient in (2.34) one finds

$$s = \pm 1, \quad C_\varphi^{(1)} = \frac{i\lambda\epsilon_n}{k + s\gamma}. \quad (2.40)$$

We can see that with this choice, the bispinor (2.34) is an eigenfunction for the projection of the total momentum along the cosmic string:

$$\hat{J}_3\psi_\sigma^{(+)} = \left(-i\partial_\phi + i\frac{q}{2}\gamma^{(1)}\gamma^{(2)}\right)\psi_\sigma^{(+)} = qj\psi_\sigma^{(+)}, \quad (2.41)$$

where

$$j = n + 1/2, \quad j = \pm 1/2, \pm 3/2, \dots \quad (2.42)$$

Hence, as a set of quantum numbers σ specifying the solutions we can take the set (λ, k, j, s) . Note that we can write the expressions for the orders of the Bessel functions in the form

$$\beta_1 = q|j| - \epsilon_j/2, \quad \beta_2 = q|j| + \epsilon_j/2. \quad (2.43)$$

For the positive frequency eigenfunctions we find the following final expression

$$\psi_\sigma^{(+)}(x) = \eta^2 C^{(+)} e^{i(qj\phi + kz)} \begin{pmatrix} H_{1/2-im\alpha}^{(1)}(\gamma\eta) J_{\beta_1}(\lambda r) e^{-iq\phi/2} \\ C_\varphi^{(1)} H_{1/2-im\alpha}^{(1)}(\gamma\eta) J_{\beta_2}(\lambda r) e^{iq\phi/2} \\ -is H_{-1/2-im\alpha}^{(1)}(\gamma\eta) J_{\beta_1}(\lambda r) e^{-iq\phi/2} \\ -is C_\varphi^{(1)} H_{-1/2-im\alpha}^{(1)}(\gamma\eta) J_{\beta_2}(\lambda r) e^{iq\phi/2} \end{pmatrix}, \quad (2.44)$$

where $C^{(+)} = C_{1\varphi}^{(1)}$ and $C_\varphi^{(1)}$ is defined by (2.40). Recall that, in this formula $\eta = \alpha e^{-t/\alpha}$. The coefficient $C^{(+)}$ in (2.44) is determined from the orthonormalization condition

$$\int d^3x \sqrt{|g|} \psi_\sigma^{(-)+} \psi_{\sigma'}^{(-)} = \delta_{\sigma\sigma'}, \quad (2.45)$$

where g is the determinant of the metric tensor corresponding to the line element (2.1). The delta symbol on the rhs of (2.45) is understood as the Kronecker delta for the discrete indices

(j, s) and as the Dirac delta function for the continuous ones (λ, k) . By using the Wronskian for the Hankel functions we find

$$|C^{(+)}|^2 = \frac{q\lambda e^{m\alpha\pi}}{32\pi\alpha^3}(\gamma + sk). \quad (2.46)$$

In a similar way, for the negative frequency eigenfunctions corresponding to the Bunch-Davies vacuum state we find the following expression

$$\psi_{\sigma}^{(-)}(x) = C^{(-)}\eta^2 e^{-i(qj\phi + kz)} \begin{pmatrix} H_{1/2-im\alpha}^{(2)}(\gamma\eta)J_{\beta_2}(\lambda r)e^{-iq\phi/2} \\ C_{\varphi}^{(2)}H_{1/2-im\alpha}^{(2)}(\gamma\eta)J_{\beta_1}(\lambda r)e^{iq\phi/2} \\ -isH_{-1/2-im\alpha}^{(2)}(\gamma\eta)J_{\beta_2}(\lambda r)e^{-iq\phi/2} \\ -isC_{\varphi}^{(2)}H_{-1/2-im\alpha}^{(2)}(\gamma\eta)J_{\beta_1}(\lambda r)e^{iq\phi/2} \end{pmatrix}, \quad (2.47)$$

with $s = \pm 1$ and

$$C_{\varphi}^{(2)} = \frac{i\epsilon_j\lambda}{k - s\gamma}, \quad |C^{(-)}|^2 = \frac{q\lambda e^{-m\alpha\pi}}{32\pi\alpha^3}(\gamma - sk). \quad (2.48)$$

For eigenfunctions (2.47) one has

$$\hat{J}_3\psi_{\sigma}^{(-)} = -qj\psi_{\sigma}^{(-)}, \quad j = \pm 1/2, \pm 3/2, \dots \quad (2.49)$$

The complete set of solutions described above may be used for the investigation of field-theoretical effects induced by the conical structure of the spacetime.

It is well-known that in the case of a scalar field in $(D + 1)$ -dimensional dS spacetime the Bunch-Davies vacuum state is not a physically realizable state for $m^2\alpha^2 \leq -D(D + 1)\xi$, where ξ is the curvature coupling parameter. In particular, this is the case for a minimally coupled massless field. The corresponding Wightman function contains infrared divergences arising from long-wavelength modes. For a fermionic field, the Bunch-Davies vacuum state is physically realizable independent of the mass.

3 Fermionic condensate

In this section we evaluate the fermionic condensate assuming that the field is prepared in the Bunch-Davies vacuum state. Having the complete set of eigenfunctions we can evaluate the corresponding vacuum expectation value (VEV) by using the mode-sum formula

$$\langle 0|\bar{\psi}\psi|0\rangle = \sum_{\sigma} \bar{\psi}_{\sigma}^{(-)}(x)\psi_{\sigma}^{(-)}(x), \quad (3.1)$$

where

$$\sum_{\sigma} = \int_{-\infty}^{+\infty} dk \int_0^{\infty} d\lambda \sum_{j=\pm 1/2, \pm 3/2, \dots} \sum_{s=\pm 1}. \quad (3.2)$$

Of course, the expression on the right of (3.1) is divergent and some renormalization procedure is needed. The important point here is that for points outside the string the local geometry of dS spacetime is not changed by the presence of the cosmic string. As a result the divergences and the renormalization procedure are the same as those in dS spacetime when the string is absent. In what follows, we implicitly assume the presence of a cutoff function which makes the mode-sum finite. By taking into account expression (2.47) for the negative frequency fermionic

eigenfunctions and introducing instead of the Hankel functions the MacDonald function $K_\nu(x)$, after some transformations we find the following expression

$$\begin{aligned} \langle 0|\bar{\psi}\psi|0\rangle &= \frac{q\eta^4}{\pi^3\alpha^3} \sum_j \int_0^\infty dk \int_0^\infty d\lambda \lambda \gamma \left[J_{qj+1/2}^2(\lambda r) + J_{qj-1/2}^2(\lambda r) \right] \\ &\times \left[|K_{1/2-im\alpha}(i\gamma\eta)|^2 - |K_{1/2+im\alpha}(i\gamma\eta)|^2 \right], \end{aligned} \quad (3.3)$$

where for the summation over j we have $\sum_j = \sum_{j=1/2, 3/2, \dots}$.

For the further transformation of this expression we use the formula

$$|K_{1/2-im\alpha}(ix)|^2 - |K_{1/2+im\alpha}(ix)|^2 = -i \left(\partial_x + \frac{1-2im\alpha}{x} \right) K_{1/2-im\alpha}(ix) K_{1/2-im\alpha}(-ix), \quad (3.4)$$

which is easily obtained on the basis of the well-known properties of the MacDonald function. This allows to present the fermionic condensate in the form

$$\begin{aligned} \langle 0|\bar{\psi}\psi|0\rangle &= \frac{q\eta^3}{i\pi^3\alpha^3} (\eta\partial_\eta + 1 - 2im\alpha) \sum_j \int_0^\infty dk \int_0^\infty d\lambda \lambda \\ &\times \left[J_{qj+1/2}^2(\lambda r) + J_{qj-1/2}^2(\lambda r) \right] K_{1/2-im\alpha}(i\gamma\eta) K_{1/2-im\alpha}(-i\gamma\eta). \end{aligned} \quad (3.5)$$

For the separate integrals with the Bessel functions we use the formula (A.6) from appendix A with the result

$$\begin{aligned} \langle 0|\bar{\psi}\psi|0\rangle &= \frac{q}{(2\pi)^{3/2}i\pi} \left(\frac{\eta}{\alpha r} \right)^3 (\eta\partial_\eta + 1 - 2im\alpha) \sum_j \int_0^\infty dx x^{1/2} \\ &\times e^{x(\eta^2/r^2-1)} \left[I_{qj+1/2}(x) + I_{qj-1/2}(x) \right] K_{1/2-im\alpha}(x\eta^2/r^2). \end{aligned} \quad (3.6)$$

By using the relation

$$(u\partial_u + 2\nu) e^{u^2} K_\nu(u^2) = 2u^2 e^{u^2} [K_\nu(u^2) - K_{\nu-1}(u^2)], \quad (3.7)$$

formula (3.6) can also be written in the form

$$\begin{aligned} \langle 0|\bar{\psi}\psi|0\rangle &= \frac{8q\alpha^{-3}}{(2\pi)^{5/2}} \sum_j \int_0^\infty dy y^{3/2} e^{y(1-r^2/\eta^2)} \text{Im} [K_{1/2-im\alpha}(y)] \\ &\times [I_{qj+1/2}(yr^2/\eta^2) + I_{qj-1/2}(yr^2/\eta^2)]. \end{aligned} \quad (3.8)$$

In particular, from this formula it follows that the fermionic condensate vanishes for a massless field. Hence, as in the case of a cosmic string in flat spacetime, the presence of the string does not break chiral symmetry of the massless theory. (In a previous publication [33], we have analyzed quantum fermionic fields in cosmic string spacetime obeying MIT bag condition on a cylindrical boundary. There we have shown that the chiral symmetry is automatically broken for massless fields.)

The expectation value (3.8) contains both contributions coming from the curvature of the de Sitter spacetime and from the non-trivial topology induced by the cosmic string. In order to extract explicitly the part induced by the string we consider the difference

$$\langle \bar{\psi}\psi \rangle_s = \langle 0|\bar{\psi}\psi|0\rangle - \langle 0|\bar{\psi}\psi|0\rangle_{\text{dS}}, \quad (3.9)$$

where $\langle 0|\bar{\psi}\psi|0\rangle_{\text{dS}}$ is the fermionic condensate in dS spacetime in the absence of the cosmic string. The formal expression for the latter is obtained from (3.8) taking $q = 1$. The corresponding

renormalization procedure using the cutoff function is described in Appendix B and the renormalized fermionic condensate is given by expression (B.4). Due to the maximal symmetry of dS spacetime and the dS invariance of the Bunch-Davies vacuum state, the corresponding VEV does not depend on the spacetime point. It vanishes for a massless field and is negative in the case of massive fermionic field.

As it was mentioned above, for points away from the string axis, the divergences do not depend on the presence of the cosmic string. Hence, the string-induced part, $\langle \bar{\psi}\psi \rangle_s$, is finite outside the string. In order to investigate this quantity we need to evaluate the difference $\sum_j [qI_{qj\pm 1/2}(x) - I_{j\pm 1/2}(x)]$. A convenient expression for this difference is provided by using the Abel-Plana summation formula in the form (see, [51, 52])

$$\sum_j f(j) = \int_0^\infty du f(u) - i \int_0^\infty du \frac{f(iu) - f(-iu)}{e^{2\pi u} + 1}. \quad (3.10)$$

From here it follows that

$$\sum_j [qf(qj) - f(j)] = -i \int_0^\infty du [f(iu) - f(-iu)] \left(\frac{1}{e^{2\pi u/q} + 1} - \frac{1}{e^{2\pi u} + 1} \right). \quad (3.11)$$

Applying this formula for the series under consideration, one finds the following integral representation:

$$\sum_j [qI_{qj+1/2}(x) + qI_{qj-1/2}(x) - I_{j+1/2}(x) - I_{j-1/2}(x)] = \frac{4}{\pi} \int_0^\infty du g(q, u) \text{Im}[K_{1/2-iu}(x)], \quad (3.12)$$

where the notation

$$g(q, u) = \cosh(u\pi) \left(\frac{1}{e^{2\pi u/q} + 1} - \frac{1}{e^{2\pi u} + 1} \right), \quad (3.13)$$

is introduced.

By making use of formula (3.12), for the correction in the fermionic condensate due to the presence of the string we find the expression

$$\begin{aligned} \langle \bar{\psi}\psi \rangle_s &= \frac{4\sqrt{2}}{\pi^{7/2}\alpha^3} \int_0^\infty du g(q, u) \int_0^\infty dy y^{3/2} e^{y(1-r^2/\eta^2)} \\ &\quad \times \text{Im}[K_{1/2-iu}(yr^2/\eta^2)] \text{Im}[K_{1/2-im\alpha}(y)]. \end{aligned} \quad (3.14)$$

The rhs of this formula is finite for points outside the string axis and the cutoff function, implicitly assumed before, may be safely removed. As it is seen from (3.14), the part in the fermionic condensate induced by the string is a function of the radial and time coordinates in the form of the ratio r/η . This property is a consequence of the maximal symmetry of dS spacetime and the Bunch-Davies vacuum state. Note that the proper distance from the axis of the string is given by $\alpha r/\eta$. Hence, the ratio r/η is the proper distance from the string measured in units of the dS curvature radius α .

Let us consider the asymptotic behavior of the fermionic condensate at small and large distances from the string. Introducing a new integration variable $x = yr^2/\eta^2$ in (3.14), we see that for points near the string the argument of the MacDonald function with the order $1/2 - im\alpha$ is large. Taking into account that for large values y and for fixed $m\alpha$ one has

$$\text{Im}[K_{1/2-im\alpha}(y)] \sim -\frac{m\alpha}{2y} \sqrt{\frac{\pi}{2y}} e^{-y}, \quad (3.15)$$

we find

$$\langle \bar{\psi}\psi \rangle_s \approx -\frac{2m\eta^2}{\pi^3\alpha^2 r^2} \int_0^\infty du g(q, u) \int_0^\infty dx e^{-x} \text{Im}[K_{1/2-iu}(x)]. \quad (3.16)$$

The integral over x is evaluated with the help of the formula [53]

$$\int_0^\infty dx x^{\beta-1} e^{-x} K_\nu(x) = \frac{\sqrt{\pi}}{2^\beta} \frac{\Gamma(\beta + \nu)\Gamma(\beta - \nu)}{\Gamma(\beta + 1/2)}. \quad (3.17)$$

After the integration over u this leads to the final result

$$\langle \bar{\psi}\psi \rangle_s \approx \frac{m(q^2 - 1)}{24\pi^2(\alpha r/\eta)^2}, \quad r/\eta \ll 1. \quad (3.18)$$

Hence, on the string axis the condensate diverges as the inverse second power of the proper distance. The part in the fermionic condensate corresponding to dS spacetime without string is constant everywhere and, hence, near the string, the condensate is dominated by the string-induced part.

In the opposite limit of large distances from the string, the main contribution to the integral over y in (3.14) comes from the region near the lower limit of the integration. By using the asymptotic formula for the MacDonald function for small values of the argument, to the leading order, we find

$$\begin{aligned} \langle \bar{\psi}\psi \rangle_s &\approx \frac{4(\eta/r)^4}{\pi^{7/2}\alpha^3} \int_0^\infty du g(q, u) \text{Im} \left[\Gamma(1/2 - im\alpha) \right. \\ &\quad \left. \times \left(\frac{\eta^2}{2r^2} \right)^{im\alpha} \int_0^\infty dx x^{1+im\alpha} e^{-x} \text{Im}[K_{1/2-iu}(x)] \right]. \end{aligned} \quad (3.19)$$

By taking into account that $2i\text{Im}[K_{1/2-iu}(x)] = K_{1/2-iu}(x) - K_{1/2+iu}(x)$, the integral over x is evaluated with the help of formula (3.17). As a result, the leading term in the asymptotic expansion of the fermionic condensate is presented in the form

$$\langle \bar{\psi}\psi \rangle_s \approx \frac{\alpha A(q, m\alpha)}{\pi^3(\alpha r/\eta)^4} \sin[2m\alpha \ln(2r/\eta) - \varphi_0], \quad r/\eta \gg 1, \quad (3.20)$$

where the function $A(q, m\alpha) \geq 0$ and the phase φ_0 are defined by the relation

$$A(q, m\alpha)e^{i\varphi_0} = \frac{\Gamma(1/2 - im\alpha)}{\Gamma(5/2 + im\alpha)} \int_0^\infty du u g(q, u) \Gamma(3/2 + im\alpha - iu) \Gamma(3/2 + im\alpha + iu). \quad (3.21)$$

Hence, at large distances the behavior of the string induced part in the fermionic condensate is damping oscillatory with the amplitude decaying as the inverse fourth power of the proper distance. The value of the ratio r/η corresponding to the first zero of the condensate increases with decreasing $m\alpha$. The same is the case for the distance between the neighbor zeros. Note that in the case of a scalar field, in dependence of the curvature radius of dS spacetime, at large distances two regimes are realized with monotonic and oscillatory behavior of the VEVs for the field squared and the energy-momentum tensor [34]. In deriving asymptotic formula (3.20) we have assumed that the value of $m\alpha$ is fixed and, hence, $m\alpha r/\eta \gg 1$. The latter means that the proper distance from the string axis is much larger than the Compton length of the spinor particle. So, at these distances we have a power-law suppression of the string-induced VEV. This is in contrast to the case of flat spacetime, in which at distances larger than the Compton length one has an exponential suppression [28].

Now let us show that in the limit $\alpha \rightarrow \infty$ and for fixed t , from formula (3.14) the expression of the fermionic condensate is obtained for the geometry of a cosmic string in Minkowski spacetime. In this limit one has $\eta \approx \alpha$. In formula (3.14) we introduce a new integration variable $x = yr^2/\alpha^2$. In the corresponding integral over x the main contribution comes from the region with $x \lesssim 1$ and for a fixed value of r the argument of the MacDonald function with the order $1/2 - im\alpha$ is large. The leading term in the asymptotic expansion of this function can be obtained by making use of the integral representation $K_\nu(z) = \int_0^\infty dt e^{-z \cosh t} \cosh(\nu t)$. For the imaginary part entering in (3.14), for fixed z and m and for large values α we find

$$\text{Im} [K_{1/2-im\alpha}(z\alpha^2)] \approx -\frac{m\sqrt{\pi}}{(2z)^{3/2}\alpha^2} e^{-m^2/2z - z\alpha^2}. \quad (3.22)$$

Using this asymptotic formula, from (3.14), in the limit $\alpha \rightarrow \infty$, for the fermionic condensate we find

$$\langle \bar{\psi}\psi \rangle_s \rightarrow \langle \bar{\psi}\psi \rangle_s^{(M)} = -\frac{2m}{\pi^3 r^2} \int_0^\infty du g(q, u) \int_0^\infty dx \text{Im}[K_{1/2-iu}(x)] e^{-m^2 r^2/2x - x}. \quad (3.23)$$

The expression in the rhs of this formula coincides with the integral representation of the fermionic condensate for the string in Minkowski spacetime found in [33].

In the left plot of figure 1 we have presented the dependence of the string-induced part in the fermionic condensate as a function of the ratio r^2/η^2 for fixed values $q = 1.5, 2$ and for $m\alpha = 2$. The right plot presents the dependence of the same quantity as a function of the field mass for $r/\eta = 2$. Recall that the ratio r/η is the proper distance from the string measured in units of the dS curvature radius α .

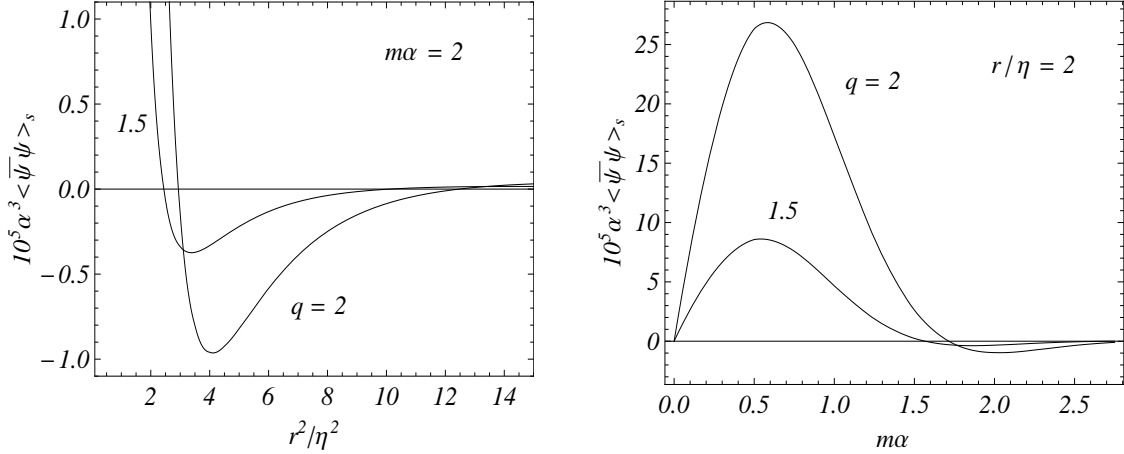


Figure 1: String-induced part in the fermionic condensate as a function of the ratio r^2/η^2 (left plot) and of the field mass (right plot.)

4 Energy-momentum tensor

Another important quantity which characterizes the properties of the quantum vacuum is the VEV of the energy-momentum tensor. In addition to describing the physical structure of the quantum field at a given point, the energy-momentum tensor acts as a source of gravity in the Einstein equations and plays an important role in modelling self-consistent dynamics involving

the gravitational field. In this section we consider the VEV of the energy-momentum tensor for the geometry of a cosmic string in dS spacetime. The corresponding VEV for a fermionic field in the geometry of a cosmic string in flat spacetime is investigated in [7, 9, 21, 22, 28]. In particular, the VEV for a massive Dirac field is considered in [28]. In a recent paper [33], the VEV of the energy-momentum tensor is analyzed for a massive spinor field obeying the MIT bag boundary condition on a cylindrical shell in the cosmic string spacetime.

The VEV of the energy-momentum tensor for the fermionic field can be evaluated by making use of the mode-sum formula

$$\langle 0|T_{\mu\nu}|0\rangle = \frac{i}{2} \sum_{\sigma} [\bar{\psi}_{\sigma}^{(-)}(x) \gamma_{(\mu} \nabla_{\nu)} \psi_{\sigma}^{(-)}(x) - (\nabla_{(\mu} \bar{\psi}_{\sigma}^{(-)}(x)) \gamma_{\nu)} \psi_{\sigma}^{(-)}(x)] , \quad (4.1)$$

with the negative frequency eigenspinors given by (2.47). The brackets in the index expressions mean the symmetrization over the indices enclosed. Note that, as before, we are working in the coordinate system (t, r, ϕ, z) . As in the case of the fermionic condensate, the presence of the cutoff function is assumed in the rhs of (4.1). Taking into account expressions (2.11) for the components of the spin connection, we see that (no summation) $\{\gamma_{\mu}, \Gamma_{\mu}\} = 0$, where the figure braces stand for the anticommutator. From here it follows that the terms in (4.1) with the spin connection will not contribute to the VEVs of diagonal components.

First we consider the vacuum energy density. Substituting the expression for the eigenfunctions from (2.47), after long but straightforward calculations, the corresponding mode-sum can be presented in the form

$$\begin{aligned} \langle 0|T_0^0|0\rangle &= -\frac{q\eta^5}{2\pi^3\alpha^4} \sum_j \int_0^{+\infty} dk \int_0^{\infty} d\lambda \lambda \left[J_{qj+1/2}^2(\lambda r) + J_{qj-1/2}^2(\lambda r) \right] \\ &\times \left[\partial_{\eta}^2 + \frac{2}{\eta} \partial_{\eta} + 4 \left(\gamma^2 + im\alpha \frac{1/2 - im\alpha}{\eta^2} \right) \right] K_{1/2-im\alpha}(i\gamma\eta) K_{1/2-im\alpha}(-i\gamma\eta). \end{aligned} \quad (4.2)$$

The rhs of this formula is expressed in terms of the functions $\mathcal{J}_{\beta}(r, \eta)$ and $\mathcal{I}_{\beta}(r, \eta)$ given in appendix A. By using the corresponding formulae we find

$$\begin{aligned} \langle 0|T_0^0|0\rangle &= -\frac{q(\eta/r)^5}{2^{1/2}\pi^{5/2}\alpha^4} \sum_j \int_0^{\infty} dx x^{3/2} e^{-x} \\ &\times [I_{qj+1/2}(x) + I_{qj-1/2}(x)] \hat{F}_u e^u K_{1/2-im\alpha}(u)|_{u=x\eta^2/r^2}, \end{aligned} \quad (4.3)$$

where we have defined the operator

$$\hat{F}_u = u\partial_u^2 + (3/2 - 2u)\partial_u - 2 + im\alpha(1/2 - im\alpha)/u. \quad (4.4)$$

Making use the properties of the MacDonald function it can be seen that

$$\hat{F}_u e^u K_{1/2-im\alpha}(u) = -\frac{1}{2} e^u [K_{1/2-im\alpha}(u) + K_{-1/2-im\alpha}(u)]. \quad (4.5)$$

This leads to the following final result for the VEV of the energy density

$$\begin{aligned} \langle 0|T_0^0|0\rangle &= \frac{4q\alpha^{-4}}{(2\pi)^{5/2}} \sum_j \int_0^{\infty} dy y^{3/2} e^{y(1-r^2/\eta^2)} \text{Re} [K_{1/2-im\alpha}(y)] \\ &\times [I_{qj+1/2}(yr^2/\eta^2) + I_{qj-1/2}(yr^2/\eta^2)]. \end{aligned} \quad (4.6)$$

As in the case of the fermionic condensate, this VEV is a function of the ratio r/η .

Now let us consider the vacuum stress along the axis of the cosmic string. From the mode-sum formula (4.1) for this component we have

$$\langle 0|T_{33}|0\rangle = (\alpha/\tau) \sum_{\sigma} k \psi_{\sigma}^{(-)+} \gamma^{(0)} \gamma^{(3)} \psi_{\sigma}^{(-)}. \quad (4.7)$$

After the substitution of the expressions for the eigenfunctions, this leads to the result

$$\begin{aligned} \langle 0|T_3^3|0\rangle &= \frac{q\eta^5}{\pi^3\alpha^4} \sum_j \int_0^\infty dk k^2 \int_0^\infty d\lambda \lambda \left[J_{qj+1/2}^2(\lambda r) + J_{qj-1/2}^2(\lambda r) \right] \\ &\times \left[K_{1/2-im\alpha}(i\gamma\eta) K_{1/2-im\alpha}(-i\gamma\eta) + K_{1/2+im\alpha}(i\gamma\eta) K_{1/2+im\alpha}(-i\gamma\eta) \right]. \end{aligned} \quad (4.8)$$

Using the integral representation (A.2) for the products of the MacDonald functions and the procedure similar to that described in Appendix A, it can be seen that

$$\langle 0|T_3^3|0\rangle = \langle 0|T_0^0|0\rangle. \quad (4.9)$$

Although this result is expected in a pure cosmic string spacetime due boost invariance of the spacetime along the z -direction, this is not the case for the present geometry. In fact, for a scalar field, the equality (4.9), in general, is not obeyed [34].

For the radial stress, from (4.1), taking into account expression (2.47), after long transformations we can see that

$$\begin{aligned} \langle 0|T_1^1|0\rangle &= \frac{q\eta^5}{\pi^3\alpha^4} \sum_j \int_0^\infty dk \int_0^\infty d\lambda \lambda^3 \left[J_{qj-1/2}(\lambda r) J'_{qj+1/2}(\lambda r) - J'_{qj-1/2}(\lambda r) J_{qj+1/2}(\lambda r) \right] \\ &\times \left[K_{1/2-im\alpha}(i\gamma\eta) K_{1/2-im\alpha}(-i\gamma\eta) + K_{1/2+im\alpha}(i\gamma\eta) K_{1/2+im\alpha}(-i\gamma\eta) \right]. \end{aligned} \quad (4.10)$$

For the further transformation we use the relation

$$\begin{aligned} &J_{qj-1/2}(\lambda r) J'_{qj+1/2}(\lambda r) - J'_{qj-1/2}(\lambda r) J_{qj+1/2}(\lambda r) \\ &= \frac{1}{\lambda^2} \left(\frac{1}{2} \partial_r^2 + \frac{1}{r} \partial_r - qj \frac{2qj-1}{r^2} \right) J_{qj-1/2}^2(\lambda r) + 2J_{qj-1/2}^2(\lambda r), \end{aligned} \quad (4.11)$$

for the Bessel function. After the changing of the order of integrations and differentiation, the part in the VEV with the first term on the rhs of (4.11) is evaluated in the way similar to that for the axial stress. The part with the second term on the right of (4.11) is evaluated similar to that described in Appendix A for the function $\mathcal{J}_\beta(r, \eta)$ with the only difference that now we need the integral $\int_0^\infty d\lambda \lambda^3 J_\beta^2(\lambda r) e^{-p\lambda^2}$. The formula for the latter is obtained from (A.4) differentiating with respect to p . Combining the results of these calculations, we obtain the following result

$$\langle 0|T_1^1|0\rangle = \frac{q(\eta/r)^5}{2^{1/2}\pi^{5/2}\alpha^4} \sum_j \int_0^\infty dx x^{3/2} e^{x\eta^2/r^2} \text{Re} \left[K_{1/2-im\alpha}(x\eta^2/r^2) \right] \hat{G}_x e^{-x} I_{qj-1/2}(x), \quad (4.12)$$

with the operator

$$\hat{G}_x = 2x\partial_x^2 + (4x+3)\partial_x + 4 - qj \frac{2qj-1}{x}. \quad (4.13)$$

Now, by using the properties of the modified Bessel function, it can be seen that

$$\hat{G}_x e^{-x} I_{qj-1/2}(x) = e^{-x} \left[I_{qj+1/2}(x) + I_{qj-1/2}(x) \right]. \quad (4.14)$$

Hence, we see that the radial stress is equal to the energy density:

$$\langle 0|T_1^1|0\rangle = \langle 0|T_0^0|0\rangle. \quad (4.15)$$

Again, in general, this property is not the case for a scalar field.

Now we turn to the azimuthal stress. The substitution of the negative frequency eigenfunctions from (2.47) to the corresponding mode-sum leads to the result

$$\begin{aligned} \langle 0|T_2^2|0\rangle &= \frac{2q^2\eta^5}{\pi^3\alpha^4 r} \sum_j j \int_0^\infty dk \int_0^\infty d\lambda \lambda^2 J_{qj-1/2}(\lambda r) J_{qj+1/2}(\lambda r) \\ &\times [K_{1/2-im\alpha}(i\gamma\eta) K_{1/2-im\alpha}(-i\gamma\eta) + K_{1/2+im\alpha}(i\gamma\eta) K_{1/2+im\alpha}(-i\gamma\eta)]. \end{aligned} \quad (4.16)$$

For the further transformation, first, we use the relation

$$\lambda J_{qj-1/2}(\lambda r) J_{qj+1/2}(\lambda r) = \left(\frac{qj-1/2}{r} - \frac{1}{2} \partial_r \right) J_{qj-1/2}^2(\lambda r). \quad (4.17)$$

Changing the order of integrations and the operator on the rhs of (4.17), the integrals can be transformed in the way similar to that used before. We come to the following result

$$\begin{aligned} \langle 0|T_2^2|0\rangle &= \frac{\sqrt{2}q^2}{\pi^{5/2}\alpha^4} \sum_j j \int_0^\infty dy y^{3/2} e^y \text{Re} [K_{1/2-im\alpha}(y)] \\ &\times \left(\frac{qj-1/2}{x} - \partial_x \right) e^{-x} I_{qj-1/2}(x) \Big|_{x=yr^2/\eta^2}. \end{aligned} \quad (4.18)$$

Taking into account that

$$\left(\frac{qj-1/2}{x} - \partial_x \right) e^{-x} I_{qj-1/2}(x) = e^{-x} [I_{qj-1/2}(x) - I_{qj+1/2}(x)], \quad (4.19)$$

we finally get

$$\begin{aligned} \langle 0|T_2^2|0\rangle &= \frac{8q^2}{(2\pi)^{5/2}\alpha^4} \sum_j j \int_0^\infty dy y^{3/2} e^{y(1-r^2/\eta^2)} \text{Re} [K_{1/2-im\alpha}(y)] \\ &\times [I_{qj-1/2}(yr^2/\eta^2) - I_{qj+1/2}(yr^2/\eta^2)]. \end{aligned} \quad (4.20)$$

It remains to consider the off-diagonal component $\langle 0|T_{01}|0\rangle$ (other components vanish by the symmetry of the problem). The corresponding mode-sum has the form

$$\langle 0|T_{01}|0\rangle = \frac{i}{4} \sum_\sigma [\psi_\sigma^{(-)+} \partial_1 \psi_\sigma^{(-)} - (\partial_1 \psi_\sigma^{(-)+}) \psi_\sigma^{(-)} + \psi_\sigma^{(-)+} \gamma^0 \gamma_1 \partial_0 \psi_\sigma^{(-)} - (\partial_0 \psi_\sigma^{(-)+}) \gamma^0 \gamma_1 \psi_\sigma^{(-)}]. \quad (4.21)$$

First of all, by using the expression for the functions $\psi_\sigma^{(-)}$ it can be seen that $\psi_\sigma^{(-)+} \partial_1 \psi_\sigma^{(-)} = (\partial_1 \psi_\sigma^{(-)+}) \psi_\sigma^{(-)}$. Second, the expression of the last two terms on the rhs of (4.21) contain the factor $C_\varphi^{(2)} + C_\varphi^{(2)*}$ which vanishes in accordance with (2.48). Hence, we conclude that the off-diagonal component $\langle 0|T_{01}|0\rangle$ vanishes. At this point we would like to say that for the case of a scalar field this component, in general, is non-zero [34]. It vanishes only for a conformally coupled massless field.

Similar to the case of the fermionic condensate, we define the subtracted VEV of the energy-momentum tensor:

$$\langle T_\mu^\nu \rangle_s = \langle 0|T_\mu^\nu|0\rangle - \langle 0|T_\mu^\nu|0\rangle_{\text{dS}}, \quad (4.22)$$

where $\langle 0|T_\mu^\nu|0\rangle_{\text{dS}}$ is the corresponding VEV in dS spacetime when the cosmic string is absent. The expressions for the components of the latter are obtained from the formulae given above

in this section taking $q = 1$. The corresponding renormalization procedure based on the introduction of the cutoff function is described in Appendix B. The renormalized VEV of the energy-momentum tensor in dS spacetime without the string is given by (B.7) and it has been previously derived in [54] (see also [55]) using the n -wave regularization method.

The explicit expressions for the string-induced parts in the energy density, axial and radial stresses are obtained using formula (3.12) (no summation over μ):

$$\begin{aligned} \langle T_\mu^\mu \rangle_s &= \frac{4\alpha^{-4}}{2^{1/2}\pi^{7/2}} \int_0^\infty dx g(q, x) \int_0^\infty dy y^{3/2} e^{y(1-r^2/\eta^2)} \\ &\quad \times \text{Im}[K_{1/2-ix}(yr^2/\eta^2)] \text{Re}[K_{1/2-im\alpha}(y)], \end{aligned} \quad (4.23)$$

with $\mu = 0, 1, 3$. In order to obtain the expression for the string-induced part in the azimuthal stress we employ the summation formula

$$\begin{aligned} \sum_j j [q^2 I_{qj+1/2}(u) - I_{j+1/2}(u) - q^2 I_{qj-1/2}(u) + I_{j-1/2}(u)] \\ = -\frac{4}{\pi} \int_0^\infty dx x g(q, x) \text{Re}[K_{1/2+ix}(u)], \end{aligned} \quad (4.24)$$

which follows from the Abel-Plana formula (3.10). This leads to the following expression

$$\begin{aligned} \langle T_2^2 \rangle_s &= \frac{8\alpha^{-4}}{2^{1/2}\pi^{7/2}} \int_0^\infty dx x g(q, x) \int_0^\infty dy y^{3/2} e^{y(1-r^2/\eta^2)} \\ &\quad \times \text{Re}[K_{1/2-ix}(yr^2/\eta^2)] \text{Re}[K_{1/2-im\alpha}(y)]. \end{aligned} \quad (4.25)$$

The string-induced parts (4.23) and (4.25) are finite at points outside the string and the renormalization is needed only for the dS part without string.

Now let us check that the string-induced parts in the VEVs obey the trace relation

$$\langle T_\mu^\mu \rangle_s = m \langle \bar{\psi} \psi \rangle_s. \quad (4.26)$$

In order to see this, we write the function in the integrand of the trace in the form

$$3\text{Im}[K_{1/2-ix}(u)] + 2x\text{Re}[K_{1/2-ix}(u)] = 2(1-u)\text{Im}[K_{1/2-ix}(u)] + 2\partial_u \text{Im}[uK_{1/2-ix}(u)]. \quad (4.27)$$

with $u = yr^2/\tau^2$. Then, in the part with the last term on the right of (4.27) we use integration by parts. In particular, from (4.26) it follows that the string-induced part in the VEV of the energy-momentum tensor is traceless for a massless fermionic field. The trace anomaly is contained in the pure dS part. In addition to (4.26), the string-induced parts obey the covariant conservation equation $\nabla_\nu \langle T_\mu^\nu \rangle_s = 0$. By taking into account that $\langle T_\mu^\nu \rangle_s$ is a function of the ratio r/η , this equation reduces to the relation

$$\langle T_2^2 \rangle_s = \partial_u (u \langle T_0^0 \rangle_s), \quad u = r/\eta. \quad (4.28)$$

This relation is easily checked making use of the formula $(\partial_u u \pm 2qj) e^{-yu^2} I_{qj\pm 1/2}(yu^2) = 0$ for the modified Bessel function.

The general formulae for the string-induced parts in the VEVs are simplified for a special case of a massless field. Noting that $K_{1/2}(y) = \sqrt{\pi/2y} e^{-y}$, for the integral over y we use the formula

$$\int_0^\infty dy y e^{-yr^2/\eta^2} K_{1/2-ix}(yr^2/\eta^2) = \frac{\pi(4x^2 + 1)}{24 \cosh(\pi x)} \frac{3 + 2ix}{(r/\eta)^4}. \quad (4.29)$$

After the evaluation of the remained integral over x , we get the result

$$\langle T_\mu^\nu \rangle_s = -\frac{(q^2 - 1)(7q^2 + 17)}{2880\pi^2(\alpha r/\eta)^4} \text{diag}(1, 1, -3, 1). \quad (4.30)$$

The massless fermionic field is conformally invariant and we could obtain this formula directly by conformal transformation of the corresponding result for the cosmic string in Minkowski spacetime [7]. Note that, in the same way, by taking into account that the electromagnetic field is conformally invariant in four-dimensional spacetime, for the corresponding string-induced part in dS spacetime, by using the result from [7], we find

$$\langle T_\mu^\nu \rangle_s^{(\text{el})} = -\frac{(q^2 - 1)(q^2 + 11)}{720\pi^2(\alpha r/\eta)^4} \text{diag}(1, 1, -3, 1). \quad (4.31)$$

Hence, with this paper we complete the investigations of the string-induced vacuum polarization for scalar, fermionic and electromagnetic fields in dS spacetime.

For a massive fermionic field the formulae for the string-induced parts in the VEV of the energy-momentum tensor are simplified at small and large distances from the cosmic string. For points near the string the main contribution to the integrals in (4.23) comes from large values of y . For these y one has $K_{1/2-i\alpha}(y) \approx K_{1/2}(y)$ and, hence, near the string, to the leading order, the string-induced parts in the VEV of the energy-momentum tensor coincide with the corresponding expressions for a massless field, given by (4.30). At large distances from the string, the main contributions to the integrals over y come from small values of y . In a way similar to that used for the case of the fermionic condensate, to the leading order we find (no summation over μ)

$$\langle T_\mu^\mu \rangle_s \approx -\frac{A(q, m\alpha)}{2\pi^3(\alpha r/\eta)^4} \cos[2m\alpha \ln(2r/\eta) - \varphi_0], \quad (4.32)$$

for $\mu = 0, 1, 3$, and

$$\langle T_2^2 \rangle_s \approx \frac{A(q, m\alpha)}{\pi^3(\alpha r/\eta)^4} \sqrt{9/4 + m^2\alpha^2} \sin[2m\alpha \ln(2r/\eta) - \varphi_0 + \varphi_1], \quad (4.33)$$

where $\varphi_1 = \arctan[3/(2m\alpha)]$. Here, the function $A(q, m\alpha)$ and the phase φ_0 are defined by formula (3.21). Now the trace relation between the asymptotics of the energy-momentum tensor and the fermionic condensate is explicitly observed. Note that the oscillations in the energy density and in the fermionic condensate are shifted by the phase $\pi/2$.

As in the case of the fermionic condensate, we can check that in the limit $\alpha \rightarrow \infty$ for fixed values of t , r and m , the VEV of the energy-momentum tensor in the geometry of a string in Minkowski spacetime is obtained. In a way similar to that used for (3.22), it can be seen that in this limit we have

$$\text{Re} [K_{1/2-i\alpha}(z\alpha^2)] \approx \frac{1}{\alpha} \sqrt{\frac{\pi}{2x}} e^{-m^2/2z - z\alpha^2}. \quad (4.34)$$

Now, with the help of this formula, in the limit under consideration one finds (no summation)

$$\begin{aligned} \langle T_\mu^\mu \rangle_s &\rightarrow \langle T_\mu^\mu \rangle_s^{(\text{M})} = \frac{2}{\pi^3 r^4} \int_0^\infty du g(q, u) \int_0^\infty dx x \text{Im}[K_{1/2-iu}(x)] e^{-m^2 r^2/2x-x}, \\ \langle T_2^2 \rangle_s &\rightarrow \langle T_2^2 \rangle_s^{(\text{M})} = \frac{4}{\pi^3 r^4} \int_0^\infty du u g(q, u) \int_0^\infty dx x \text{Re} [K_{1/2-iu}(x)] e^{-m^2 r^2/2x-x}, \end{aligned} \quad (4.35)$$

with $\mu = 0, 1, 3$. Again, these expressions coincide with the integral representations given in [33] (with the missprint in the expression for the energy density corrected).

In figure 2 we have plotted the string induced parts in the VEVs of the energy-momentum tensor components as functions of r^2/η^2 for fixed values $q = 1.5, 2$ and for $m\alpha = 2$. The left/right plot corresponds to the energy density/azimuthal stress. Figure 3 presents the same quantities as functions of $m\alpha$ for $r/\eta = 2$.

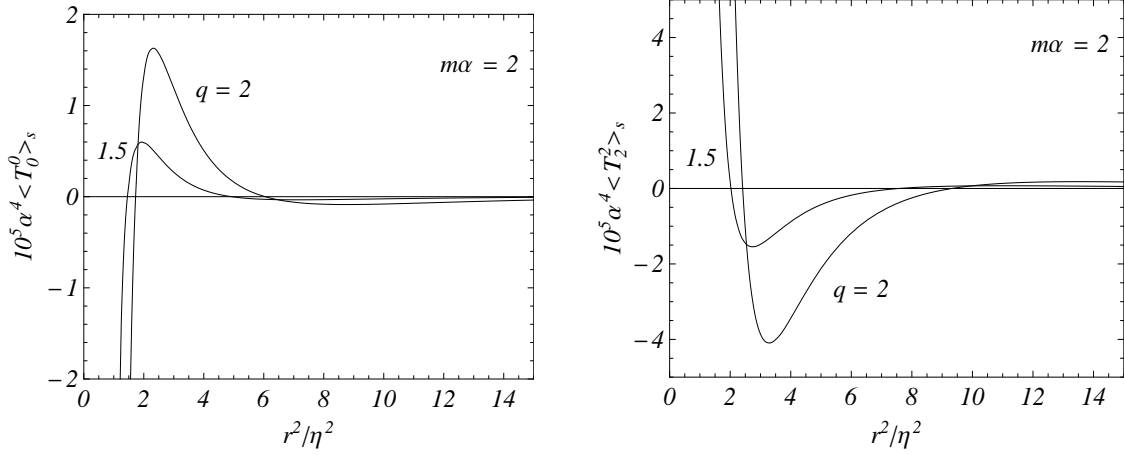


Figure 2: String-induced parts in the energy density (left plot) and the azimuthal stress (right plot) as functions of the ratio r^2/η^2 for fixed values $q = 1.5, 2$ and for $m\alpha = 2$.

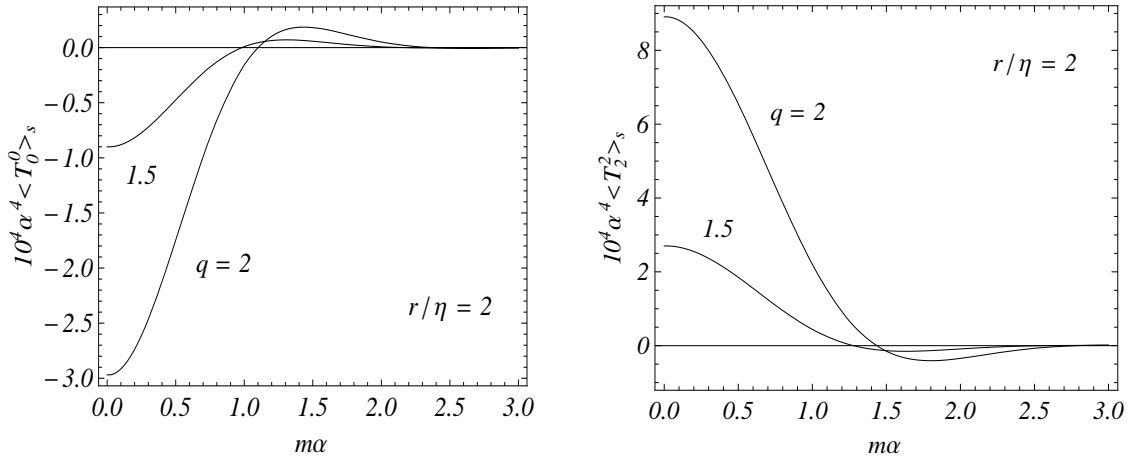


Figure 3: String-induced parts in the energy density (left plot) and the azimuthal stress (right plot) as functions of the field mass for $r/\eta = 2$.

5 Conclusion

In the present paper we have investigated the polarization of the fermionic vacuum induced by a cosmic string in dS spacetime. We have assumed that the field is prepared in the Bunch-Davies vacuum state. Unlike to the case of a scalar field, for a fermionic field the Bunch-Davies vacuum is a physically realizable state independent of the field mass. Among the most important covariant characteristics of the vacuum state are the fermionic condensate and the expectation value of the

energy-momentum tensor. In order to evaluate these quantities we have used the direct mode-summation method. In this approach the complete set of normalized eigenfunctions for the Dirac equation is used. In section 2 we have found these eigenfunction for the geometry under consideration. They are given by expressions (2.44) and (2.47) for the positive and negative frequency solutions, respectively.

On the basis of these expressions, in section 3 we have investigated the fermionic condensate. The corresponding mode-sum is given by expression (3.8). This expression contains both contributions coming from the curvature of the de Sitter spacetime and from the non-trivial topology induced by the cosmic string. In order to extract explicitly the part induced by the string, we have subtracted the term corresponding to the geometry of dS spacetime when the string is absent. Due to the maximal symmetry of dS spacetime and the dS invariance of the Bunch-Davies vacuum state, the corresponding VEV does not depend on the spacetime point. Using the Abel-Plana summation formula, the string-induced part in the fermionic condensate is presented in the form (3.14). This part vanishes for a massless field. As the presence of the cosmic string does not change the local geometry outside the string axis, the renormalization in the calculation of the corresponding condensate is reduced to that for dS spacetime without the string.

The general formula for the string-induced part in the fermionic condensate is simplified in asymptotic regions of small and large distances from the axis of the string. For points near the string the leading term in the corresponding asymptotic expansion is given by expression (3.18) and the condensate diverges as the inverse square of the proper distance. The part in the fermionic condensate corresponding to dS spacetime without string is constant everywhere and near the string, the condensate is dominated by the string-induced part. At large distances from the string the behavior of the string-induced part in the fermionic condensate is described by formula (3.20) and this behavior is damping oscillatory with the amplitude decaying with the inverse fourth power of the distance. The value of the proper distance corresponding to the first zero of the condensate increases with decreasing $m\alpha$. The same is the case for the distance between the neighbor zeros. As an additional check, we have shown that in the limit of large dS curvature radius the result is recovered for the fermionic condensate in the geometry of a cosmic string on background of flat spacetime.

The VEV of the energy-momentum tensor for a fermionic field is investigated in section 4. Unlike to the case of a scalar field, this VEV is diagonal. The axial and radial stresses are equal to the energy density. The corresponding mode-sums are transformed to the form (4.6), for the energy density, and to the form (4.20), for the azimuthal stress. Similar to the case of the fermionic condensate, we have extracted from the VEVs the parts corresponding to dS spacetime without the string. For the subtracted part, integral representations are obtained useful for the numerical evaluation. For a massless field the string-induced part in the VEV of the energy-momentum tensor is given by a simple formula (4.30) and is related to the corresponding flat spacetime result by the standard conformal transformation. In this case the energy density and stresses are monotonic functions of the distance from the string. This is not the case for a massive field. The corresponding VEVs are given by formulae (4.23) and (4.25). These VEVs depend on the radial and time coordinates in the combination r/η which is the proper distance from the string in units of the dS curvature radius. For points near the cosmic string, the leading term in the corresponding asymptotic expansion does not depend on the field mass and the string-induced part diverges as the inverse fourth power of the distance. At large distances, the behavior of the string-induced parts is damping oscillatory. As for the case of the condensate, the amplitude of these oscillations decays with the inverse fourth power of the distance. The oscillations in the energy density and in the fermionic condensate are shifted by the phase $\pi/2$. The power-law decay of the string-induced VEVs at large distances from the string is in contrast

to the case of a string in flat spacetime, in which at distances larger than the Compton length of a spinor particle one has an exponential suppression.

The presence of a cosmic string in the period of inflation will lead to the modification of the power-spectrum for vacuum fluctuations. The cosmic string breaks the homogeneity of dS space and, as a result, the power-spectrum depends on the distance from the string. In the case of a scalar field this dependence was investigated in [40], where it has been shown that the influence of the string appears in the form of a universal multiplier in the power-spectrum which does not depend on time, on the field mass and on the curvature coupling parameter. An interesting topic for the further investigation is the influence of the fermionic fluctuations, discussed in the present paper, on the power-spectrum from the inflation. The latter is imprinted in the anisotropies of the cosmic microwave background radiation and the observational data on these anisotropies could constrain the density and the parameters of the cosmic strings at the end of inflation.

The present paper completes the investigations of the string-induced vacuum polarization for scalar, fermionic and electromagnetic fields in dS spacetime. The case of scalar field was considered in [34] and for the electromagnetic field the corresponding VEV of the energy-momentum tensor is obtained from the flat spacetime result by a simple conformal transformation (see (4.31)).

Acknowledgments

E.R.B.M. thanks Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and FAPES-ES/CNPq (PRONEX) for partial financial support. A.A.S. was supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and by the Armenian Ministry of Education and Science Grant No. 119.

A Integral representation

In this appendix we consider the transformation of the expression

$$\mathcal{J}_\beta(r, \eta) = \int_0^\infty dk \int_0^\infty d\lambda \lambda J_\beta^2(\lambda r) K_\nu(i\gamma\eta) K_\nu(-i\gamma\eta). \quad (\text{A.1})$$

As the first step we use the following integral representation for the product of the MacDonald functions [56]:

$$K_\nu(i\gamma\eta) K_\nu(-i\gamma\eta) = \int_0^\infty du u^{-1} \int_0^\infty dy \cosh(2\nu y) \exp[u(\gamma\eta)^2 - (\gamma\eta)^2 u \cosh(2y) - 1/2u]. \quad (\text{A.2})$$

Substituting this in (A.1), the integral over k is taken explicitly:

$$\begin{aligned} \mathcal{J}_\beta(r, \eta) &= \frac{\sqrt{\pi}}{2^{3/2}\eta} \int_0^\infty d\lambda \lambda J_\beta^2(\lambda r) \int_0^\infty du u^{-3/2} \int_0^\infty dy \\ &\quad \times \frac{\cosh(2\nu y)}{\sinh y} \exp[-2u(\lambda\eta)^2 \sinh^2 y - 1/2u]. \end{aligned} \quad (\text{A.3})$$

As the next step, for the λ -integral we use the formula [53]

$$\int_0^\infty d\lambda \lambda J_\beta^2(\lambda r) e^{-p\lambda^2} = \frac{e^{-r^2/2p}}{2p} I_\beta(r^2/2p). \quad (\text{A.4})$$

After introducing a new integration variable $x = r^2/(4u\eta^2 \sinh^2 y)$, we obtain

$$\mathcal{J}_\beta(r, \eta) = \frac{\pi^{1/2}}{2^{1/2}r^3} \int_0^\infty dy \cosh(2\nu y) \int_0^\infty dx x^{1/2} e^{-x} I_\beta(x) \exp(-2x\eta^2 \sinh^2 y/r^2). \quad (\text{A.5})$$

Changing the order of integrations, the integral over y is expressed in terms of the MacDonald function and we obtain the following formula

$$\mathcal{J}_\beta(r, \eta) = \frac{\sqrt{\pi}}{2^{3/2}r^3} \int_0^\infty dx x^{1/2} e^{x(\eta^2/r^2-1)} I_\beta(x) K_\nu(x\eta^2/r^2). \quad (\text{A.6})$$

Next, we consider the integral

$$\mathcal{I}_\beta(r, \eta) = \int_0^\infty dk \int_0^\infty d\lambda \lambda \gamma^2 J_\beta^2(\lambda r) K_\nu(i\gamma\eta) K_\nu(-i\gamma\eta). \quad (\text{A.7})$$

By taking into account (A.2), in a way similar to that for the function $\mathcal{J}_\beta(r, \eta)$, it can be seen that

$$\mathcal{I}_\beta(r, \eta) = -\frac{\sqrt{\pi/2}}{r^5} \partial_{\eta^2} \eta^2 \int_0^\infty dx x^{3/2} e^{x(\eta^2/r^2-1)} I_\beta(x) K_\nu(x\eta^2/r^2). \quad (\text{A.8})$$

Formulae (A.6) and (A.8) are used in the main text in order to derive integral representations for string-induced parts in the fermionic condensate and in the VEV of the energy-momentum tensor.

B Fermionic vacuum densities in dS spacetime without string

In this section we consider the VEVs in dS spacetime when the string is absent. The corresponding energy-momentum tensor is investigated in [54] by using the n -wave regularization method. Here we show that our approach based on the cutoff function method leads to the same result. In addition, we will evaluate the fermionic condensate.

First we consider the fermionic condensate. The corresponding integral representation is obtained from (3.8) by taking $q = 1$. In this case the series over j is summed by using the formula $I_0(x) + 2 \sum_{n=1}^\infty I_n(x) = e^x$. Introducing explicitly the exponential cutoff, we find

$$\langle 0 | \bar{\psi} \psi | 0 \rangle_{\text{dS}}^{(\beta)} = \frac{8\alpha^{-3}}{(2\pi)^{5/2}} \int_0^\infty dy y^{3/2} e^{(1-\beta)y} \text{Im} [K_{1/2-i\alpha}(y)], \quad (\text{B.1})$$

where $\beta > 0$ is a cutoff parameter. The integral in this formula is expressed in terms of the associated Legendre function of the first kind (see [53]). We will write the corresponding result in terms of the hypergeometric function:

$$\int_0^\infty dy y^{3/2} e^{(1-\beta)y} K_{1/2-i\alpha}(y) = -\frac{i\pi\sqrt{\pi/2}}{\sinh(\pi m\alpha)} \partial_\beta^2 F(i\alpha, -i\alpha + 1; 1; 1 - \beta/2). \quad (\text{B.2})$$

A convenient expansion in powers of β for the hypergeometric function on the right hand side of this formula is given in [57]. For the fermionic condensate this leads to the expansion

$$\langle 0 | \bar{\psi} \psi | 0 \rangle_{\text{dS}}^{(\beta)} = -\frac{m\alpha}{4\pi^2\alpha^3} \left\{ \frac{2}{\beta} + (1 + m^2\alpha^2) [\ln(\beta/2) + 2\text{Re} \psi(i\alpha) - 2\ln(m\alpha) + b_1] + o(\beta) \right\}, \quad (\text{B.3})$$

where $\psi(x)$ is the digamma function. The presence of the logarithmic term in this expansion signifies about the renormalization non uniqueness in the form of b_1 . This non uniqueness

can be removed by imposing an additional renormalization condition. As such a condition we will require that $\langle \bar{\psi}\psi \rangle_{\text{dS,ren}} \rightarrow 0$ in the limit $m \rightarrow \infty$ (for the discussion of this condition in the context of the Casimir effect see [58]). With this condition for the renormalized fermionic condensate in dS spacetime we find

$$\langle \bar{\psi}\psi \rangle_{\text{dS,ren}} = \frac{m}{2\pi^2\alpha^2} [(1 + m^2\alpha^2) [\ln(m\alpha) - \text{Re} \psi(im\alpha)] + 1/12]. \quad (\text{B.4})$$

It can be checked that expression (B.4) coincides with the result obtained from the effective Lagrangian with the help of formula $\langle \bar{\psi}\psi \rangle_{\text{dS,ren}} = -(2/\sqrt{|g|})\partial_m L_{\text{eff}}$. The expression for the effective Lagrangian in dS spacetime is derived in [59] by using the dimensional regularization. For large masses, $m\alpha \gg 1$, in the leading order we have

$$\langle \bar{\psi}\psi \rangle_{\text{dS,ren}} \approx -\frac{11}{480\pi^2\alpha^4 m}. \quad (\text{B.5})$$

For a massless field the fermionic condensate in dS spacetime vanishes. In the limit $m\alpha \ll 1$, for the leading term one has $\langle \bar{\psi}\psi \rangle_{\text{dS,ren}} \approx m \ln(m\alpha)/(2\pi^2\alpha^2)$. The numerical evaluation shows that $\langle \bar{\psi}\psi \rangle_{\text{dS,ren}} < 0$ for a massive fermionic field with the minimum at $m\alpha \approx 0.27$.

Now we turn to the energy-momentum tensor. Taking $q = 1$ in formulae (4.6) and (4.20) we find

$$\langle 0|T_\mu^\nu|0 \rangle_{\text{dS}}^{(\beta)} = \frac{4\alpha^{-4}\delta_\mu^\nu}{(2\pi)^{5/2}} \int_0^\infty dy y^{3/2} e^{(1-\beta)y} \text{Re} [K_{1/2-im\alpha}(y)]. \quad (\text{B.6})$$

As we could expect from the maximal symmetry of dS spacetime, all components coincide. By using formula (B.2), expanding the hypergeometric function and imposing the renormalization condition $\langle T_\mu^\nu \rangle_{\text{ren}} \rightarrow 0$ for $m \rightarrow \infty$, for the renormalized VEV of the energy-momentum tensor we find

$$\langle T_\mu^\nu \rangle_{\text{ren}} = \frac{\alpha^{-4}\delta_\mu^\nu}{8\pi^2} \left\{ m^2\alpha^2(1 + m^2\alpha^2) [\ln(m\alpha) - \text{Re} \psi(im\alpha)] + \frac{m^2\alpha^2}{12} + \frac{11}{120} \right\}. \quad (\text{B.7})$$

This expression coincides with the one derived in [54] by using the n -wave regularization method. In the massless limit, Eq. (B.7) gives the well-known expression for the trace anomaly. The energy density corresponding to (B.7) has the maximum for a massless field, becomes zero at $m\alpha \approx 0.467$ and is negative for larger values of this parameter. For large values of the field mass, $m\alpha \gg 1$, to the leading order we have

$$\langle T_\mu^\nu \rangle_{\text{ren}} \approx -\frac{\delta_\mu^\nu \alpha^{-6}}{960\pi^2 m^2}, \quad (\text{B.8})$$

and the VEV vanishes, in accordance with renormalization condition imposed.

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